

Attempt all questions :

1. Let f be a smooth vectorfield on \mathbb{R}^n . Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function such that $DV(x)f(x) < 0$ at all points x when $f(x) \neq 0$. Prove that
 - (a) V is strictly decreasing along each nonconstant trajectory of the system $\dot{x} = f(x)$.
 - (b) For each trajectory Γ of the system $\dot{x} = f(x)$, $\omega(\Gamma)$ consists of only equilibrium points (assuming $\omega(\Gamma)$ is nonempty). [5+5]
2. (a) Show that the origin of the system

$$\begin{aligned}
 \dot{x}_1 &= -x_2 - x_1x_2^2 + x_3^2 - x_1^3 \\
 \dot{x}_2 &= x_1 + x_3^3 - x_2^3 \\
 \dot{x}_3 &= -x_1x_3 - x_3x_1^2 - x_2x_3^2 - x_3^5
 \end{aligned}$$

is an asymptotically stable equilibrium point.

- (b) Consider the linearization $\dot{x} = Df(0)x$ of the above problem. Prove that the origin is a stable equilibrium point. Is it asymptotically stable? [10]
3. Show that for the linear system $\dot{x} = Ax$ where $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ at any point $(x_0, 0)$ on the x -axis, the Poincare map for the focus at the origin is given by $P(x_0) = x_0 \exp(\frac{2\pi a}{|b|})$. [10]
4. (a) Consider the system

$$\begin{aligned}
 \dot{x} &= -y + x(r^4 - 3r^2 + 1) & \text{where } r^2 &= x^2 + y^2 \\
 \dot{y} &= x + y(r^4 - 3r^2 + 1)
 \end{aligned}$$

Show that there is a periodic orbit of their system in the annular region $A = \{x \in \mathbb{R}^2 / 1 < |x| < 2\}$

- (b) Let f be a smooth vectorfield on \mathbb{R}^2 . A point $x \in \mathbb{R}^2$ is called a recurrent point if there exists a sequence $t_n \nearrow \infty$ such that the sequence $\phi_{t_n}(x)$ converges to x (here ϕ_t is the flow associated to f). Prove that if x is a nonequilibrium recurrent point then the trajectory through x of the system $\dot{x} = f(x)$ is periodic. [5+5]
5. Let T be the torus, let $D \subseteq T$ be a disk and $S = T - \text{Int}(D)$. Observe that the boundary of S , ∂S , is equal to the boundary of D , ∂D , which is a circle. Consider a smooth vectorfield f on S which is tangent to ∂S . Show that f has a critical point. [10]